國立高雄大學九十八學年度碩士班招生考試試題

科目：線性代數
考試時間：100 分鐘

系所：
應用數學系
本科原始成績：100 分

是否使用計算機：否

1. (20%) Write down the definition of the following terms.

(1) $\beta = \{v_1, \cdots, v_n\}$ is a **basis** of the vector space V

(2) λ is an **eigenvalue** of the matrix A

(3) T is a **linear transformation**

(4) Matrices A, B are **similar**

(5) **Rank-Nullity theorem**

2. (10%) True or false? Just write down the answer. No need to prove it.

1. The rank of any upper triangular matrix is the number of nonzero entries on its diagonal.
2. Let A, B, C be matrices. If A commutes with B，and B commutes with C，then A commutes with C.
3. If V, W are subspaces of R^n，then $V \cup W$ is a subspace of R^n.
4. If a real matrix A has only the eigenvalues 1 and -1，then A must be orthogonal.
5. If A is a positive definite matrix, then the largest entry of A must be on the diagonal.

3. (30%)

(1) Let $A = \begin{bmatrix} k & 1 & 1 & 1 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \\ 1 & 1 & 1 & k \end{bmatrix}$. If rank(A)=3，find k.

(2) Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. Suppose $A^2 - AB = I_3$，Find B.

(3) Let $\beta = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and $\mu = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \right\}$ be two bases of the subspace V in R^3.

Find the change of the basis matrix from β to μ.

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(4) Given a basis $\beta = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \right\}$，apply Gram-Schmidt process to find an orthonormal basis.

(5) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, for the quadratic form $q(\vec{x}) = 6x_1^{\ 2} - 7x_1x_2 + 8x_2^{\ 2}$, find a symmetric matrix A such that $q(\vec{x}) = \vec{x} \cdot A \cdot \vec{x}$

4. (10%) Suppose a, b, c are real numbers. Discuss the solution of the system of the equations in terms of a, b, c.

$$\begin{cases} x + 2y + az = 1 \\ 3x + 4y + bz = -1 \\ 2x + 10y + 7z = c \end{cases}$$

5. (10%) It is known that $A = \begin{bmatrix} -2 & 0 & 0 \\ 2 & a & 2 \\ 3 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & b \end{bmatrix}$ are similar matrices. Find the matrix P such that $P^{-1}AP = B$.

6. (10%) An $n \times n$ matrix A is called *nilpotent* if $A^m = 0$ for some positive integer m. Let A be a nilpotent matrix and choose the smallest m such that $A^m = 0$. Let $\vec{v}$ in R^n such that $A^{m-1}\vec{v} \neq 0$. Show that the vectors $\vec{v}, A\vec{v}, A^2\vec{v}, \cdots, A^{m-1}\vec{v}$ are linearly independent.

7. (10%) Let A be a real $n \times n$ <u>symmetric</u> matrix, P be an $n \times n$ invertible matrix. Let α be the eigenvector of A corresponding to the eigenvalue λ. Find the eigenvector of $(P^{-1}AP)^T$ corresponding to the eigenvalue λ.

背面尚有試題

國立高雄大學九十八學年度碩士班招生考試試題

科目：高等微積分
考試時間：100 分鐘

系所：
應用數學系
本科原始成績：100 分

是否使用計算機：否

1. (50%) Prove or disprove the following propositions.
   (1.1) Let $\mathcal{F}$ be the family of sets consisting of all subsets of natural number. Then $\mathcal{F}$ is uncountable.
   (1.2) Every infinite sequence of real numbers has a convergent subsequence.
   (1.3) Let $\{a_n\}$ be a sequence of real numbers. If a_n has only one accumulation point $a \in \mathbb{R}$. Then a_n converges to a.
   (1.4) Let f a continuous function defined on $\mathbb{R}$ and G an open set in $\mathbb{R}$. Then the set $f(G)$ is always open.
   (1.5) Every closed set in $\mathbb{R}$ is the intersection of a countable collection of open sets.

2. (15%) Let $f(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x}\rangle$ be a function defined on $\mathbb{R}^3$, where $\mathbf{x} \in \mathbb{R}^3$,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

   and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^3$.
   (a) Compute $Df(\mathbf{x})$ and $D^2 f(\mathbf{x})$.
   (b) Determine whether f has a local extremal or saddle point.

3. (10%) Evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

4. (10%) Evaluate the integral

$$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx.$$

5. Let

$$f(x) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

   (a) (8%) Compute $f_x(0, 0)$, $f_y(0, 0)$.
   (b) (7%) Is f differentiable at $(0, 0)$? Justify your answer.